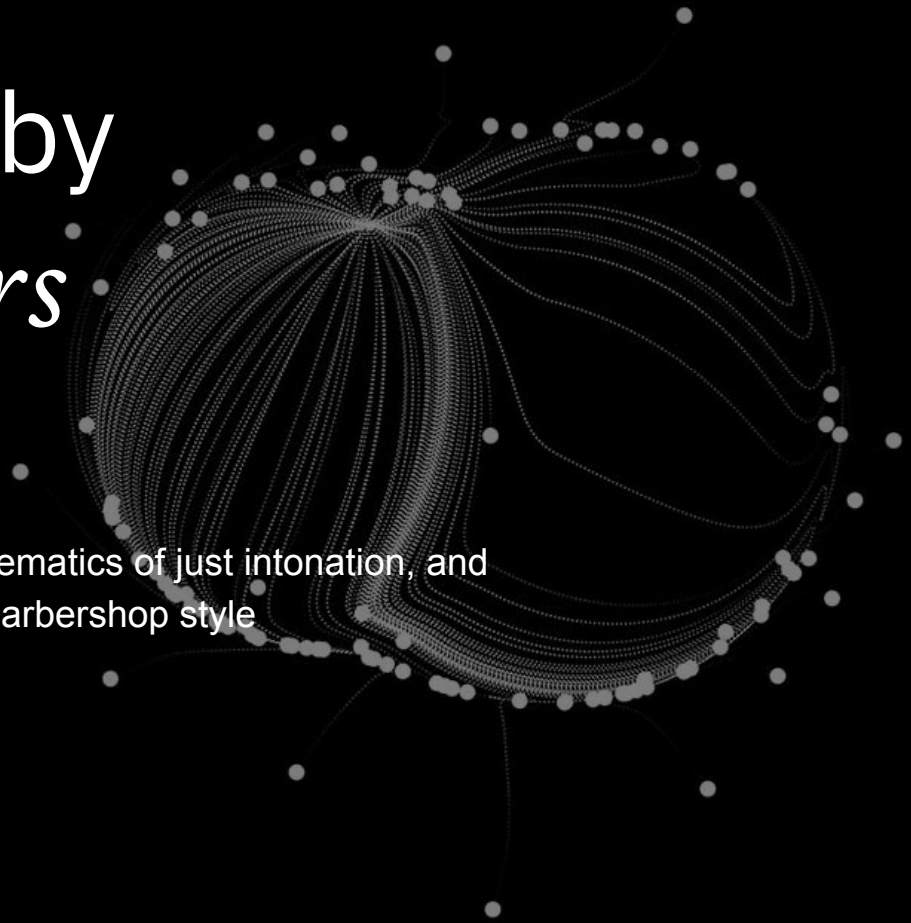
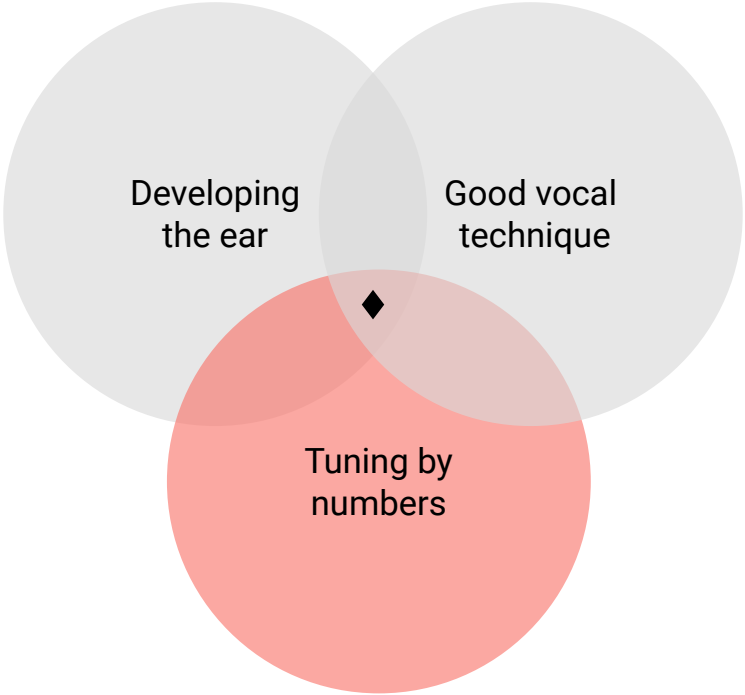
Tuning by *Numbers*



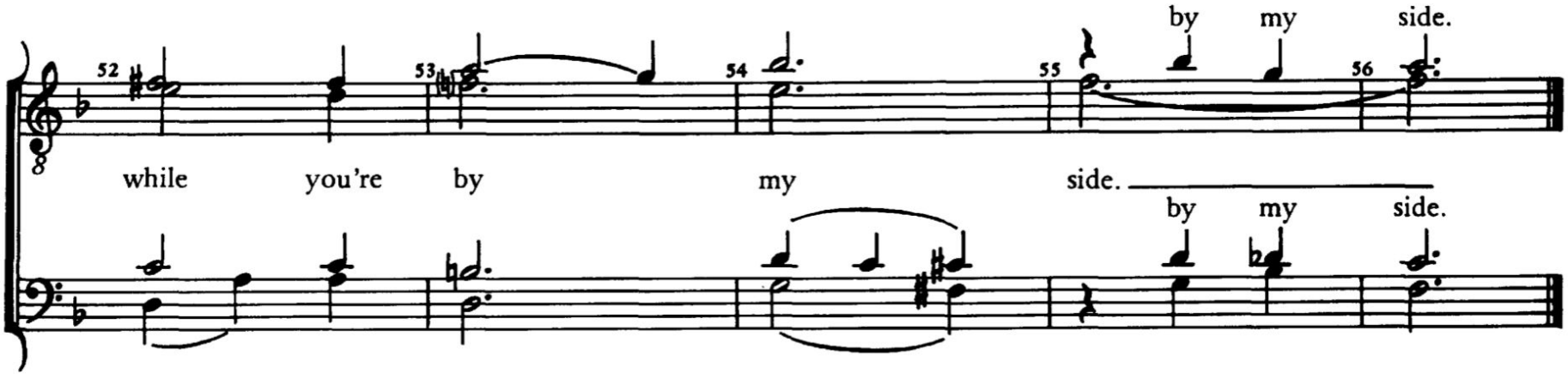
- The physics and mathematics of just intonation, and how to apply it in the barbershop style

HARMONY COLLEGE 2019

There are many aspects to singing in tune!

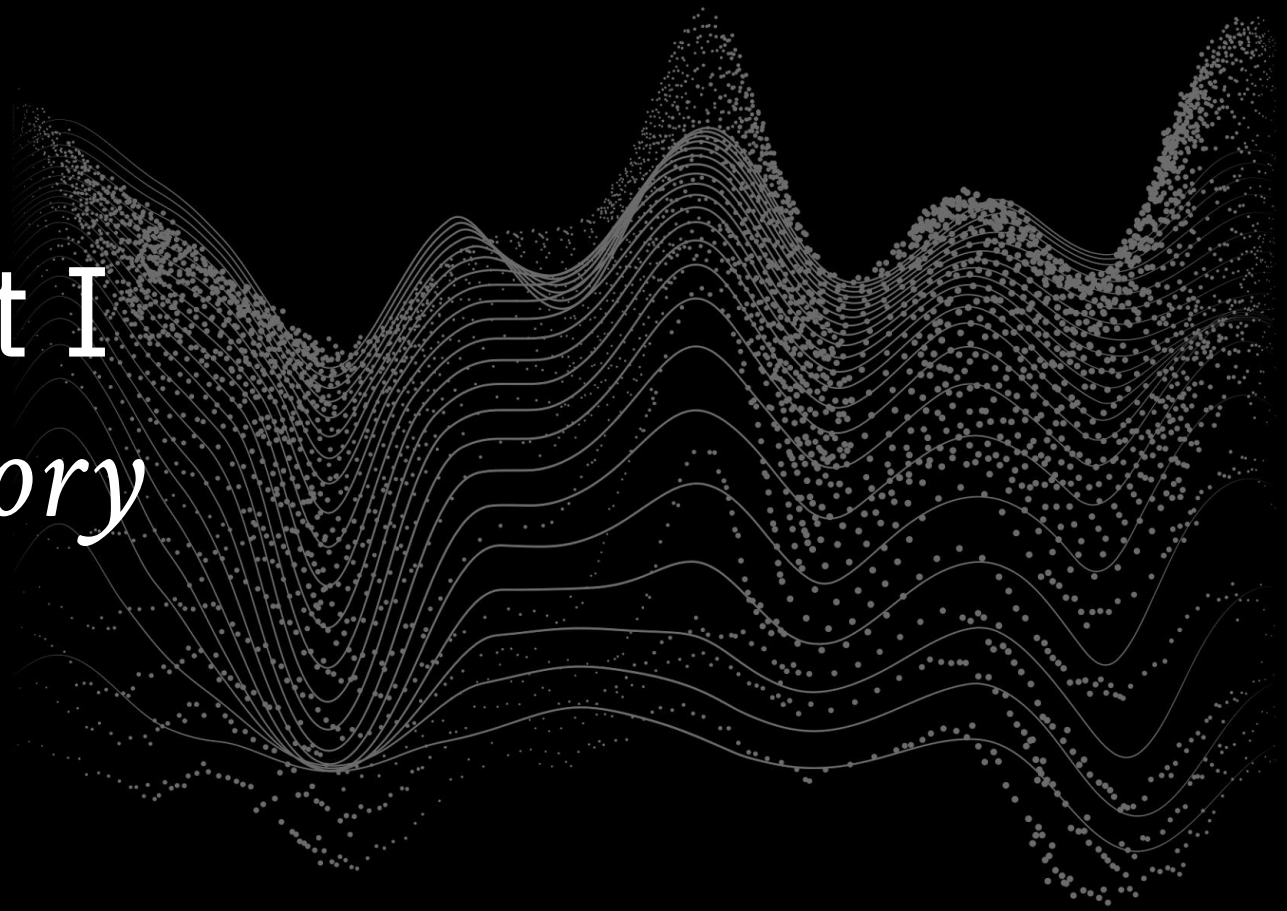


Sweet and Lovely - *A problem we all face*



Contents

	PART 1 - Theory
01	Introduction to the harmonic series and Just Intonation
02	Equal Temperament
03	Justly tuning whole chords
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06	Sweet and Lovely - the solution!



Part I
Theory

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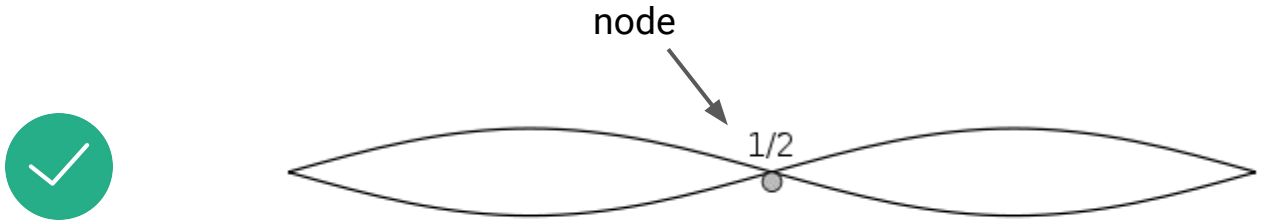
Tuning by Numbers

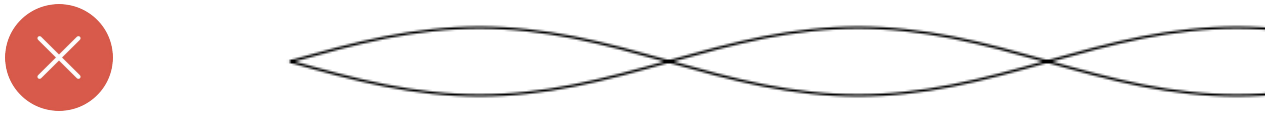
1. Introduction to the harmonic series and Just Intonation

Imagine a vibrating string:



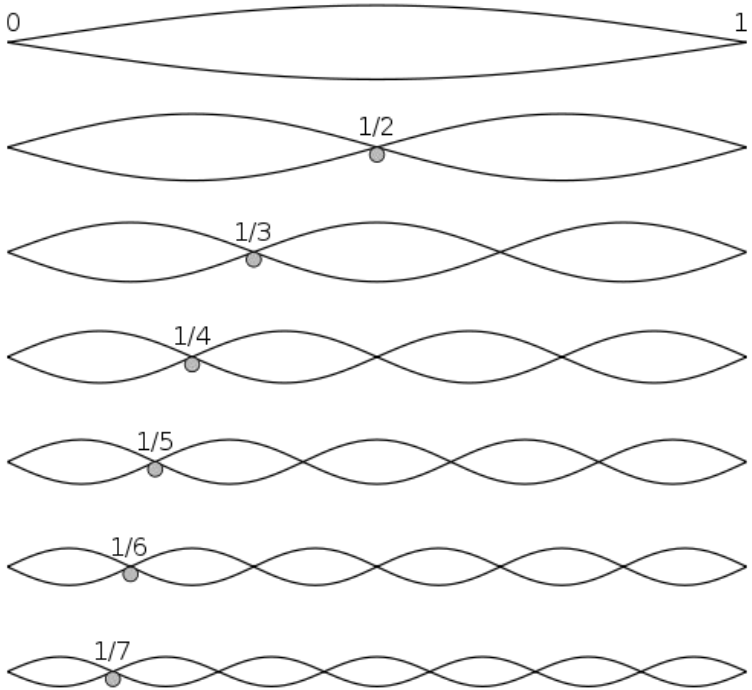
We can introduce “nodes” to split the string into equal parts





?

This results in a particular set of frequencies we call a “Harmonic Series”



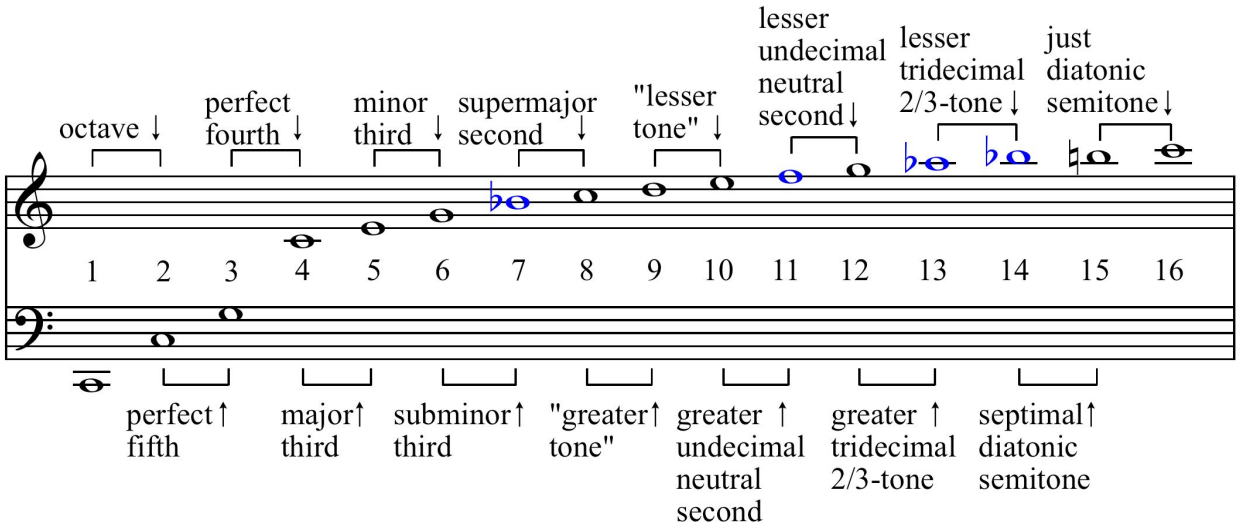
“The 1st harmonic”

“The 2nd harmonic”

...

“The 5th harmonic”

This results in a particular set of frequencies we call a “Harmonic Series”



Blue notes are a long way from equal temperament

Just Intonation = Tuning chords according to the harmonic series



A method of tuning

A “frequency ratio” is a comparison of the frequencies of the two pitches in an interval

246Hz



164 Hz

A “frequency ratio” is a comparison of the frequencies of the two pitches in an interval

246Hz



164 Hz

246 : 164

3 : 2

“1.5”

A “frequency ratio” is a comparison of the frequencies of the two pitches in an interval

246Hz

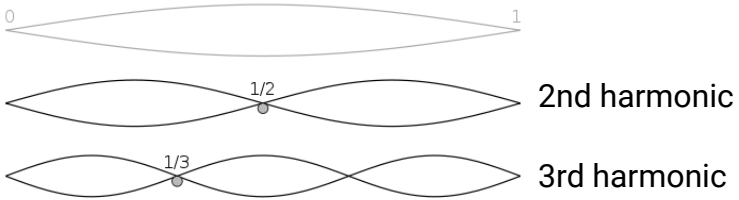


164 Hz

246 : 164

3 : 2

“1.5”



We can write a table like so:

Interval	Example notes	Frequency ratio
Octave	*C - C*	1:2
Perfect fifth	*C - G*	**2:3**
Perfect fourth	*G - C*	3:4
Major third	*C - E*	4:5
Minor third (major 3rd - 5th)	*E - G*	5:6
Minor third (*5th - 7th*)	*G - Bb*	6:7
Tone (*7th - 8ve*)	*Bb - C*	7:8
Tone (*8ve - 9th*)	*C - D*	8:9
Tone (*9th - 3rd*)	*D - E*	9:10

We can write a table like so:

Interval	Example notes	Frequency ratio
More exotic ratios:		
Major third (*7th - 9th*)	*Bb - D*	7:9
Major sixth	*C - A*	3:5
Minor sixth	*E - C*	5:8
Minor (dominant) seventh	*C - Bb*	4:7
Major seventh	*C - B*	8:15
Tritone (small: *3rd - 7th*)	*E - Bb*	5:7
Tritone (large: *7th - 3rd*)	*Bb - E*	7:10
Major 9th	*C - D*	4:9

There are two major thirds and two minor thirds

Interval	Example notes	Frequency ratio
Major third (tonic - major 3rd)	*C - E*	4:5
Major third (*7th - 9th*)	*Bb - D*	7:9
Minor third (major 3rd - 5th)	*E - G*	5:6
Minor third (*5th - 7th*)	*G - Bb*	6:7

There are two major thirds and two minor thirds

Interval	Example notes	Frequency ratio
Major third (tonic - major 3rd)	*C - E*	4:5
Major third (*7th - 9th*)	*Bb - D*	7:9
Minor third (major 3rd - 5th)	*E - G*	5:6
Minor third (*5th - 7th*)	*G - Bb*	6:7

And, by the way:

$$\frac{5}{4} = 1.25 \qquad \frac{9}{7} = 1.285\ldots$$

Hello My Baby (Ringmasters tag) - *Two different major 3rds*



F9 F9 Ab maj7

Tuning by Numbers

2. Equal Temperament

Video: equal temperament and just intonation

Equal temperament - *some quick facts*

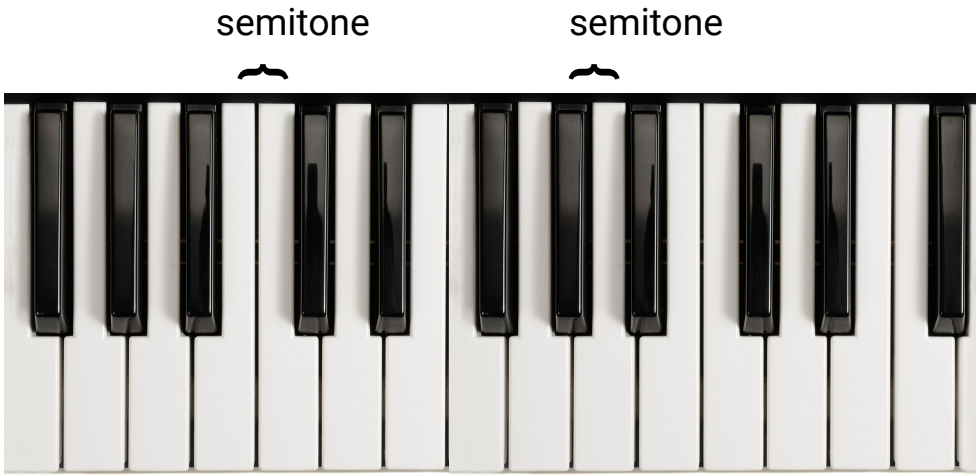
- Evil
- Cunning trick of the devil
- Subversion of the one true tuning
- Should be rejected at every turn



Equal temperament - *deserving villain?*

...Maybe not

Equal temperament - *what's the deal?*



Under *ET*, all semitones are equal

Why do *ET* and *JI* not agree?

Interval	Example notes	Frequency ratio
Octave	*C - C*	1:2

Why do *ET* and *JI* not agree?

Interval	Example notes	Frequency ratio
Major third	*C - E*	4:5



$$\frac{5}{4} \times \frac{5}{4} \times \frac{5}{4} = 1.953125$$

Why do *ET* and *JI* not agree?

Interval	Example notes	Frequency ratio
Minor third	*E - G*	5:6



$$\frac{6}{5} \times \frac{6}{5} \times \frac{6}{5} \times \frac{6}{5} = 2.0736$$

Why do *ET* and *JI* not agree?

Interval	Example notes	Frequency ratio
Perfect fourth	*G - C*	3:4



…



$$\frac{4}{3} \times \frac{4}{3} \times \quad \cdots \quad \times \frac{4}{3} \times \frac{4}{3} = 1.97^{*}$$

*actually (4/3)^12) = 31.57, but several octaves up. Dividing by 16 (descending 4 octaves) gives 1.97.

Video: What is a cent?

Table of cent adjustments for common barbershop intervals (1/2)

Interval	Example notes	Frequency ratio	Adjustment to achieve Just Intonation (cents)
Octave	*C - C*	1:2	0
Perfect Fifth	*C - G*	2:3	2
Perfect Fourth	*G - C*	3:4	-2
Major Third	*C - E*	4:5	-14
Minor Third	*E - G*	5:6	16
Minor Third (*5th - 7th*)	*G - Bb*	6:7	-33
Tone (*7th - 8ve*)	*Bb - C*	7:8	31
Tone (*8ve - 9th*)	*C - D*	8:9	4
Tone (*9th - 3rd*)	*D - E*	9:10	-18

Table of cent adjustments for common barbershop intervals (2/2)

Interval	Example notes	Frequency ratio	Adjustment to achieve Just Intonation (cents)
More exotic ratios:			
Major 3rd (*7th - 9th*)	*Bb - D*	7:9	35
Major sixth	*C - A*	3:5	-16
Minor sixth	*E - C*	5:8	14
Minor (dominant) seventh	*C - Bb*	4:7	-31
Major seventh	*C - B*	8:15	12
Tritone (small: *3rd - 7th*)	*E - Bb*	5:7	-17
Tritone (large: *7th - 3rd*)	*Bb - E*	7:10	17
Major 9th	*C - D*	4:9	4

Conclusion: Equal temperament disagrees with just intonation





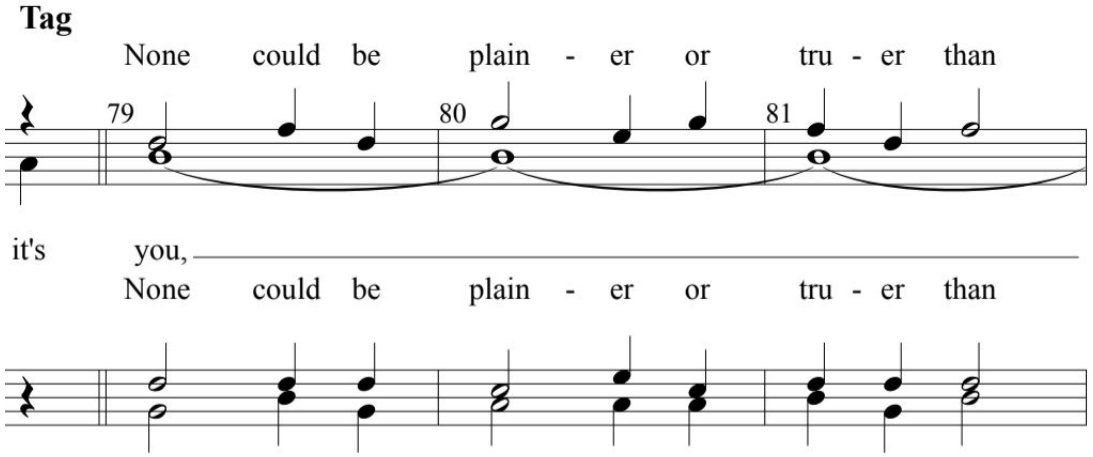
(But that’s okay)

Tuning by Numbers

3. Justly tuning whole chords

It's You -

Moving around a post





B G7

It's You -

voice swapping between different tunings



Tuning by Numbers

3. Justly tuning whole chords

The aim: make micro-adjustments so that the ratios between notes are as *simple* as possible

2 : 3



Simple

20 : 31

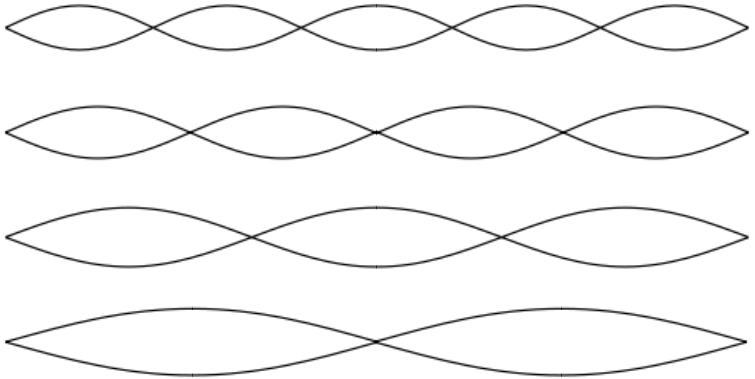


Not simple!

Most common barbershop chords have *simple* relationships (and this is a good thing)



Ab major

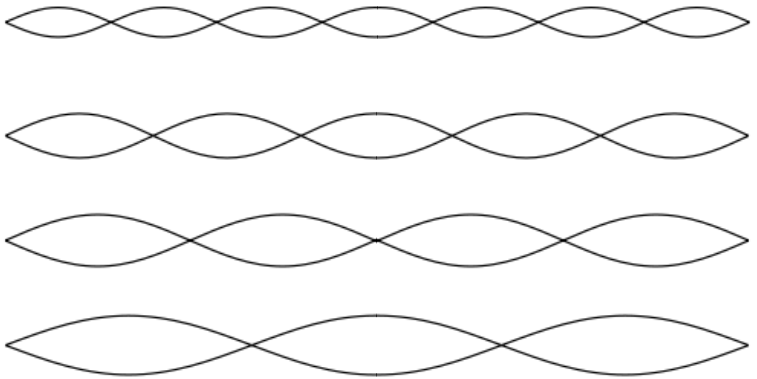


Simple!

Most common barbershop chords have *simple* relationships (and this is a good thing)



Ab 7



Also simple!

Major 6 and minor 7 chords have the same notes

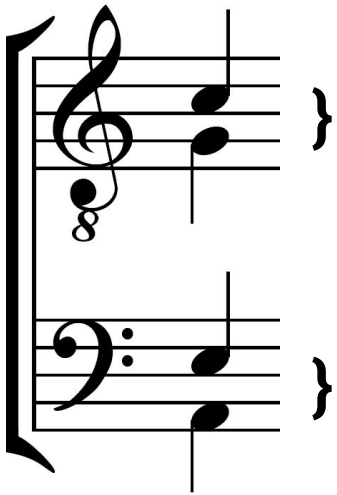
C major 6

A minor 7



...but the simplest ratios, and thus the microtuning, are different

C major 6		A minor 7
16 18	4th	-33 -31
2 0	5th	2 0



The half-diminished / minor 6 chord

Examples:



*I Won't Send Roses: "still while there's a fighting chance, just turn and **go**"*



*That Old Quartet of Mine: "just to sing that song just one more **time**"*

The half-diminished / minor 6 chord



How do we tune it?



Observations:

- Perfect 5th between C and G
- Tritone between Eb and A
- Lowest three notes make a minor triad

We can try a *traditional* minor triad tuning



?

2

+16

0

Observations:

- Perfect 5th between C and G
- Tritone between Eb and A
- Lowest three notes make a minor triad

What if we try an *alternative* minor triad tuning?



?

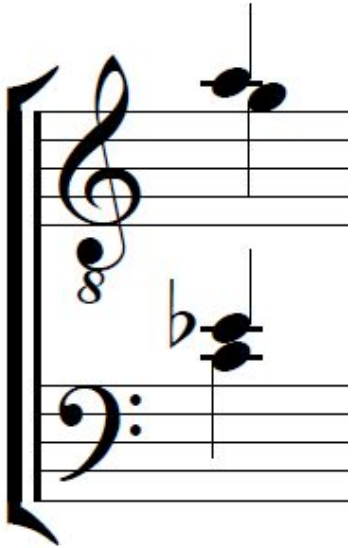
2

-33

0

Interval	Frequency ratio	Adjustment (cents)
Minor third *(3rd - 5th)*	5:6	16
Minor third (*5th - 7th*)	6:7	-33

Add in tuning for the A♮ a (large) tritone away from the Eb

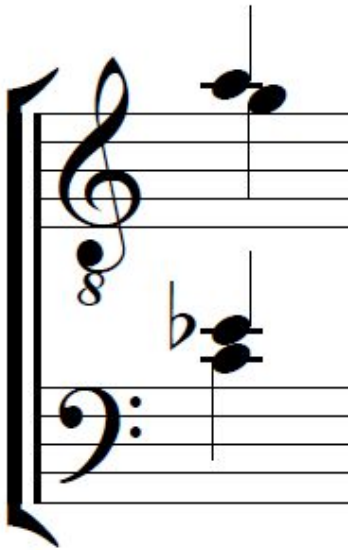


-16
2

-33
0

Interval	Frequency ratio	Adjustment (cents)
Tritone (small: *3rd - 7th*)	5:7	-17
Tritone (large: *7th - 3rd*)	7:10	17

Shift all numbers up by 2 cents - *are these more familiar?*



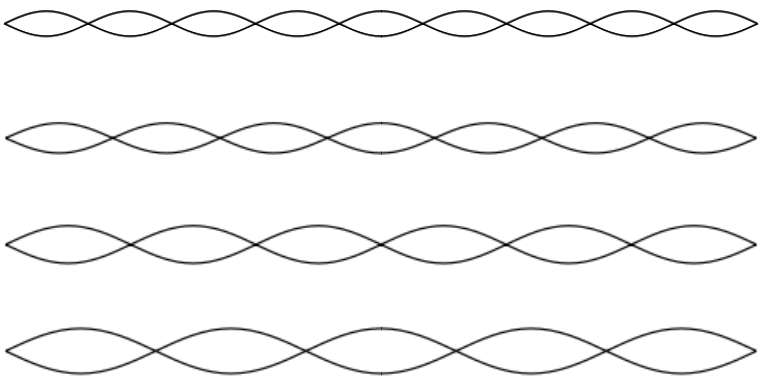
-16	-14
2	4
-33	-31
0	2

Shift all numbers up by 2 cents - *are these more familiar?*



-14	3rd
4	9th
-31	7th
2	5th
0	TONIC!

The half-diminished / minor 6 / rootless 9 chord



5 : 6 : 7 : 9



Part II

Putting it into practice

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PART 1 - Theory

01 Introduction to the harmonic series and Just Intonation

02 Equal Temperament

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04 Tuning around the lead

05 Step-by-step: working out the numbers

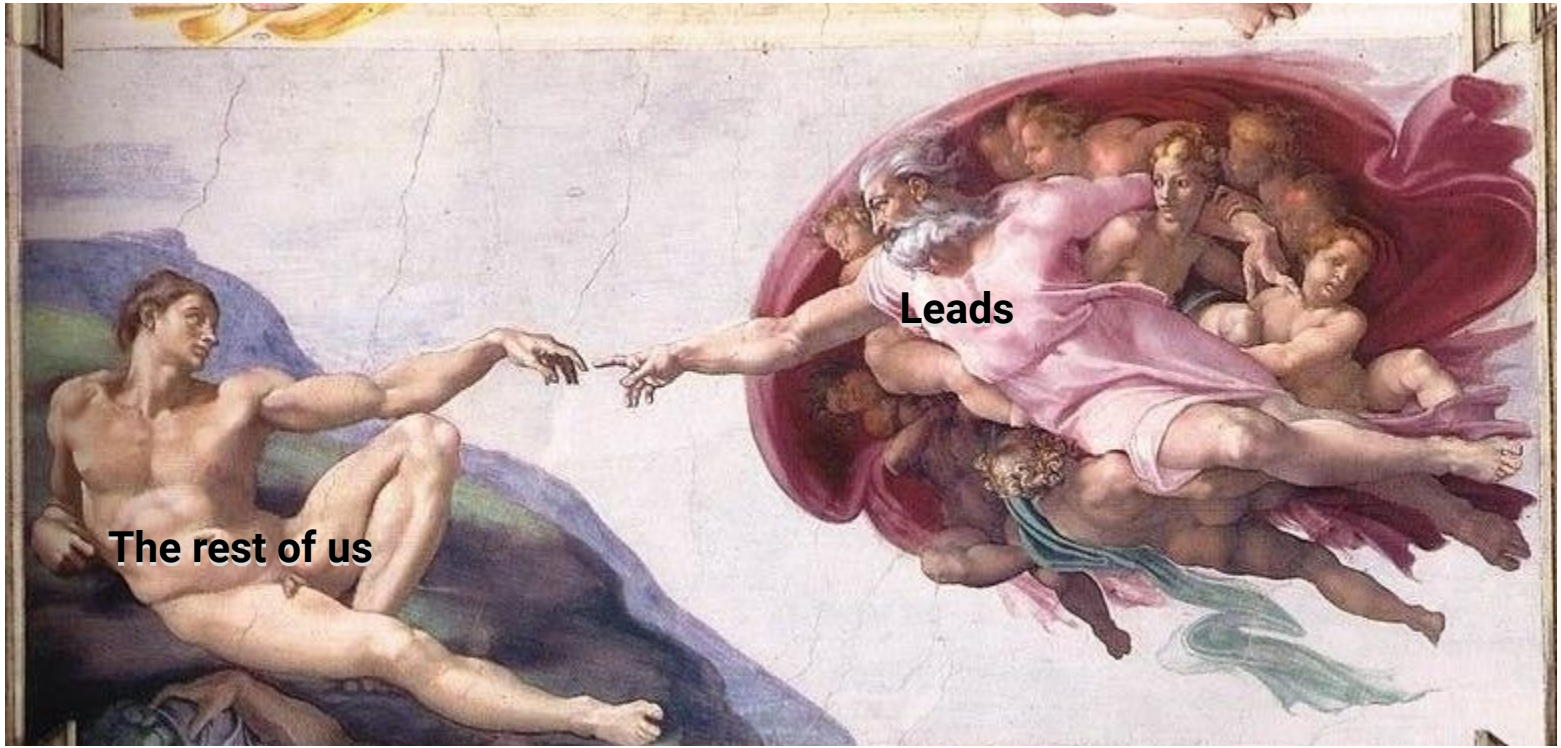
06 Sweet and Lovely - the solution!

Tuning by Numbers

4. Tuning around the lead

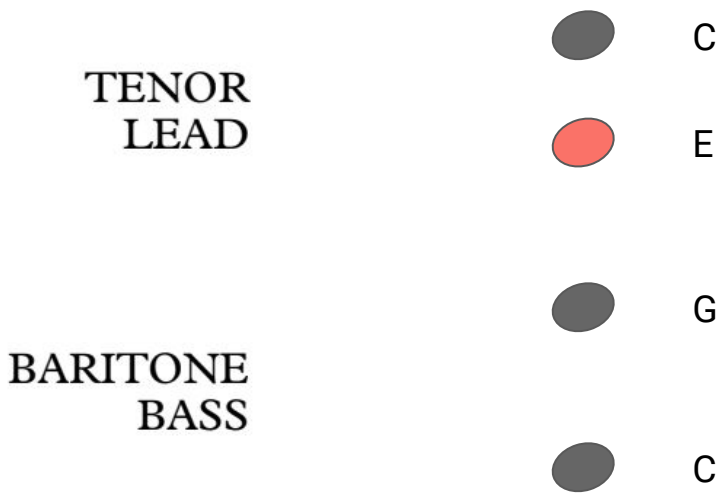
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The “Lead is God” theory



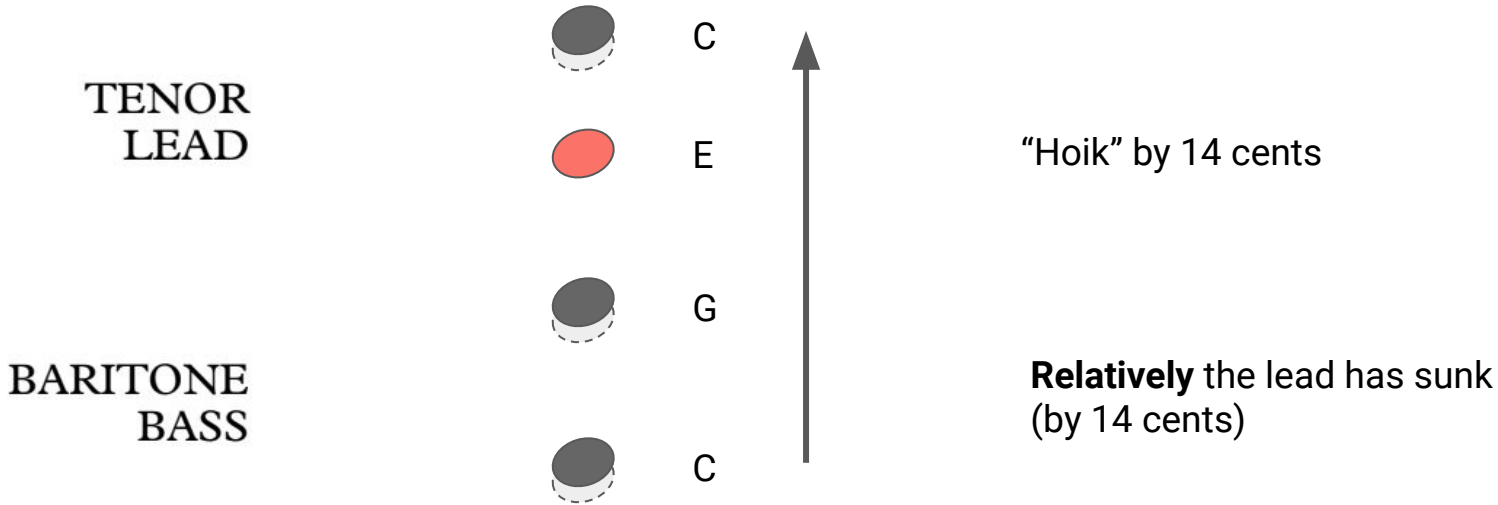
The harmony parts must “flatten” and “hoik” in order to accommodate the lead

C major chord



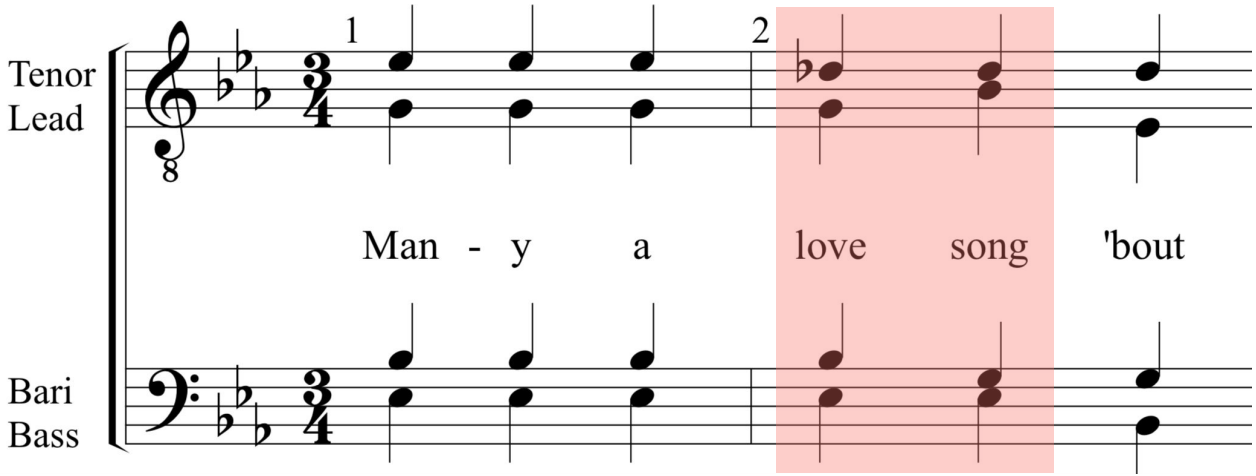
The harmony parts must “flatten” and “hoik” in order to accommodate the lead

C major chord



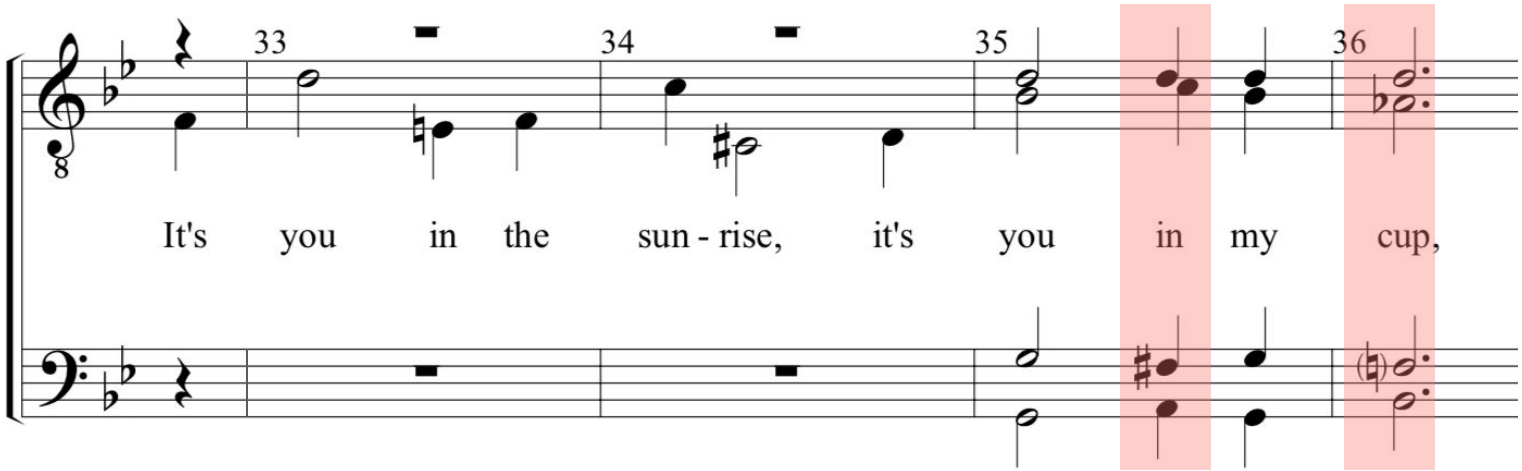
There are exceptions to “Lead is God”

Exception 1: voice swapping



There are exceptions to “Lead is God”

Exception 2: extreme adjustment

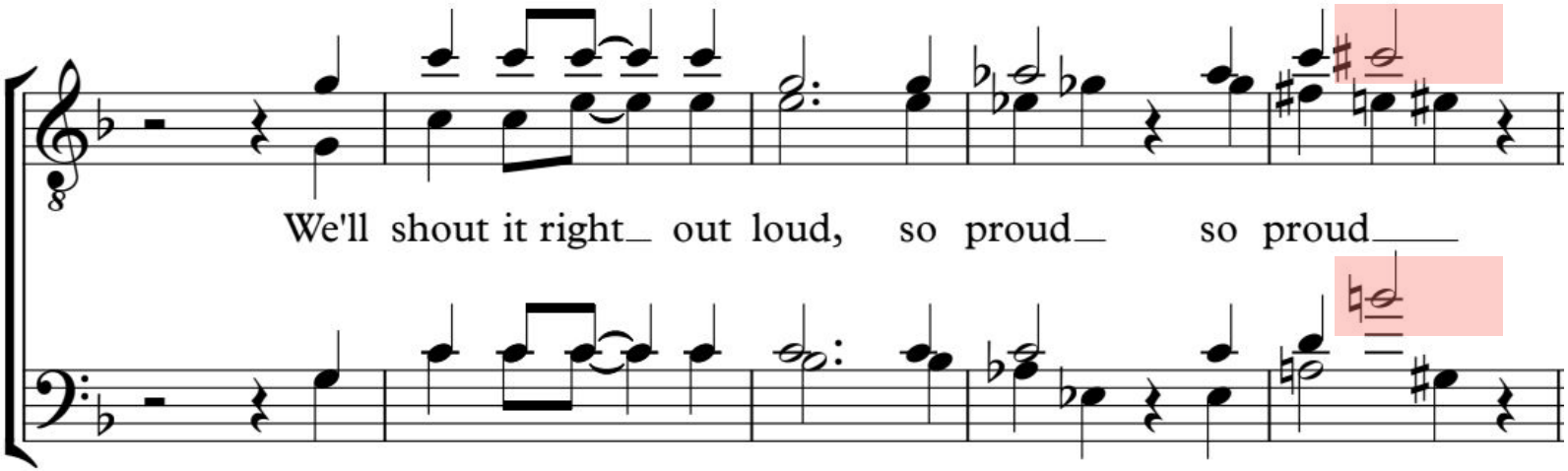


There are exceptions to “Lead is God”

Exception 3: posts or pedals (held notes)

There are exceptions to “Lead is God”

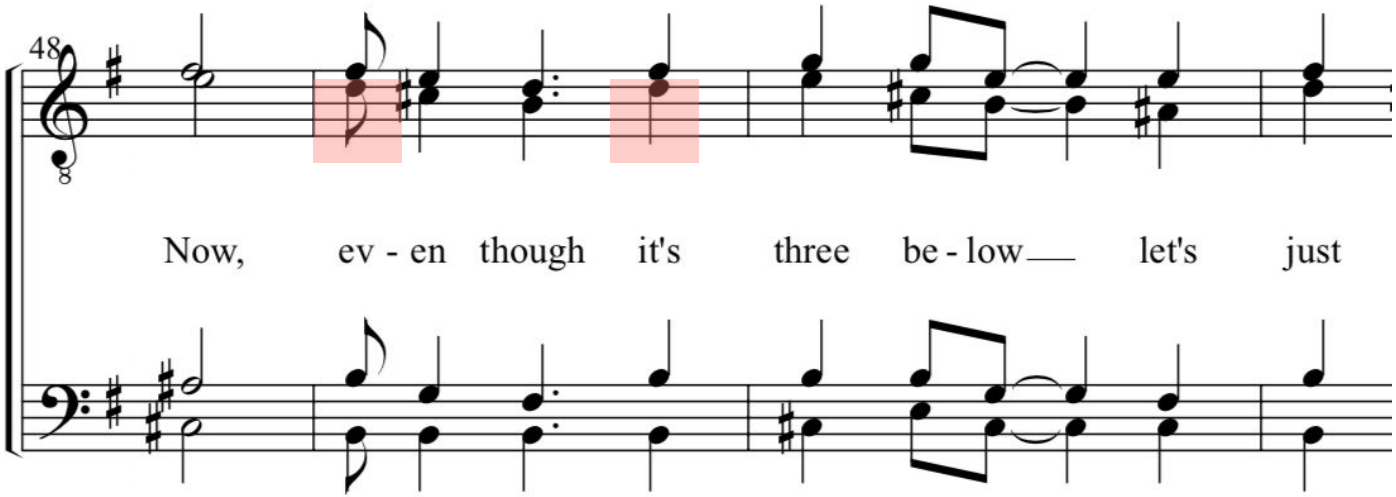
Exception 3: posts or pedals (held notes)



A9 C#7

There are exceptions to “Lead is God”

Exception 4: “a feeling of going flat”



There are exceptions to “Lead is God”

Exception 5: Low major triads where the lead is on the third



5. Step-by-step: working out the numbers

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Step 1: Identify chord (and voicing)



Step 1: Identify chord (and voicing)



7th
3rd

D dominant 7

Tonic
5th

Step 2: Apply voicing cent adjustments to each note



7th	- 31
3rd	- 14

D dominant 7

Tonic	0
5th	2

Step 3: Decide which note to designate the “zero note”



7th	- 31
3rd	- 14

D dominant 7

Tonic	0
5th	2

Ordinarily the lead, but we should take into account potential exceptions to the rule

Step 4: Subtract the cent adjustment of the “zero note” from all other notes



7th	- 31	- 31 + 14
3rd	- 14	0

D dominant 7

Tonic	0	0 + 14
5th	2	2 + 14

Step 5: Write in cent adjustments

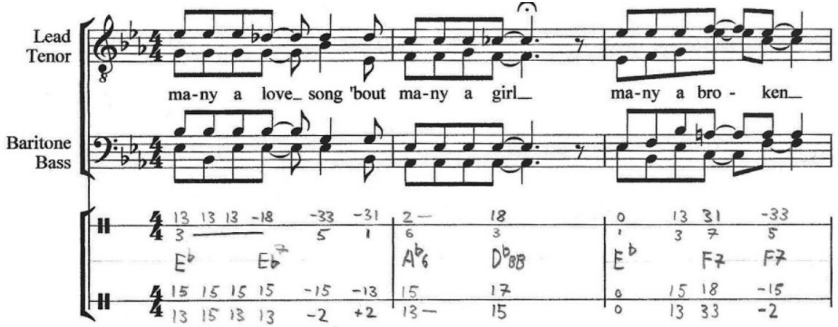


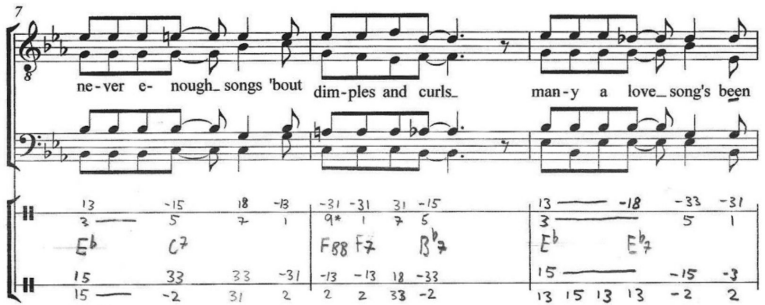
7th	- 31	- 31 + 14	- 17
3rd	- 14	0	0

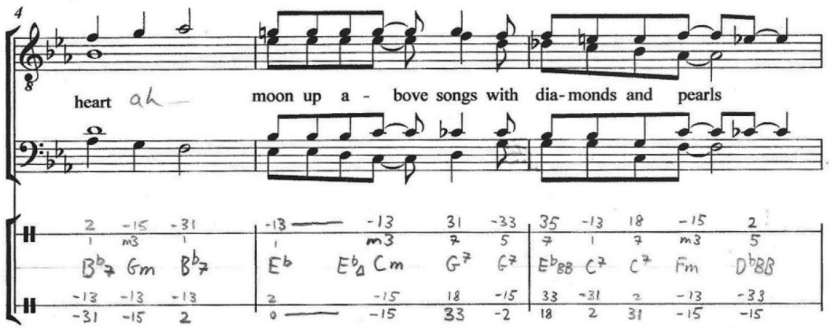
D dominant 7

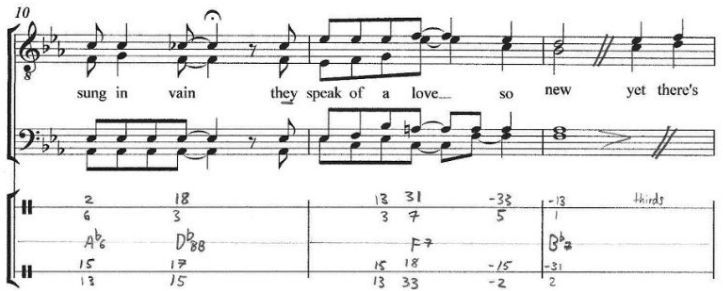
Tonic	0	0 + 14	14
5th	2	2 + 14	16

This is what “taking it too far” looks like:

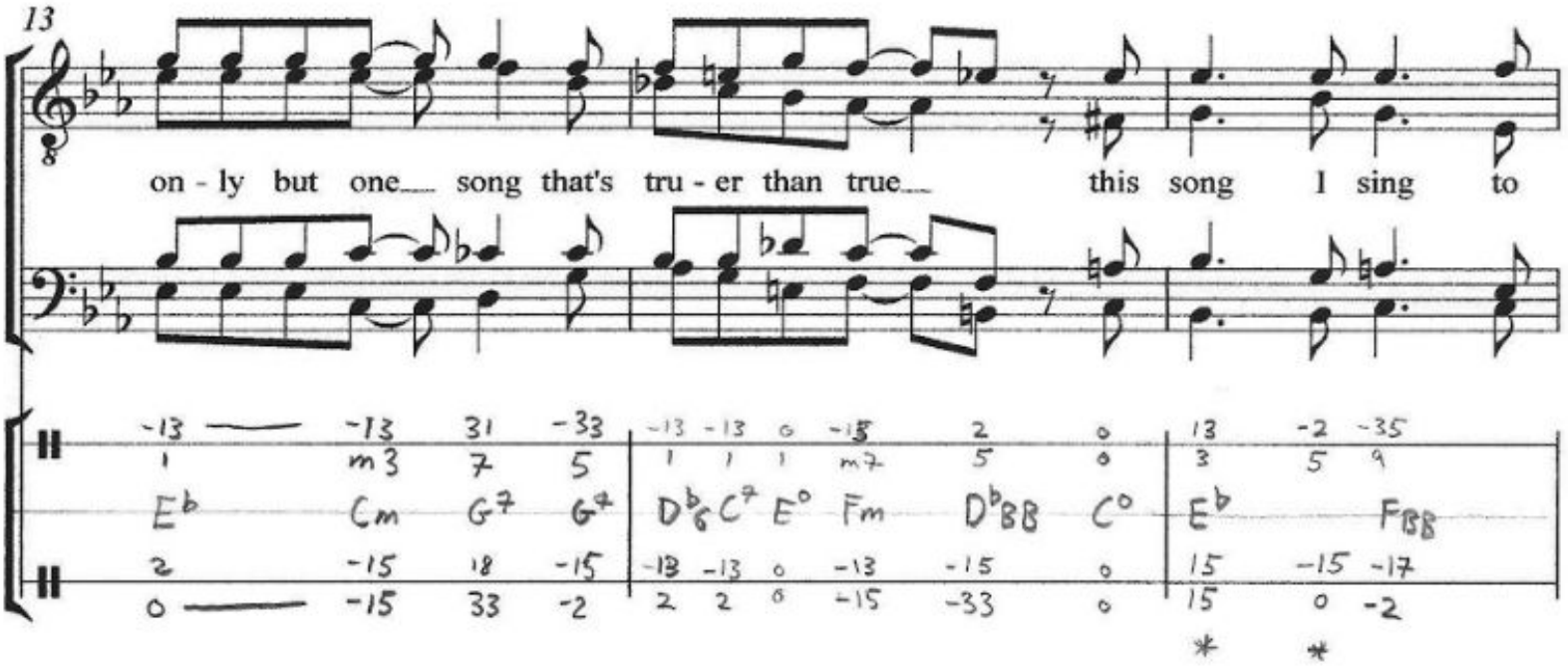








This is what “taking it too far” looks like:

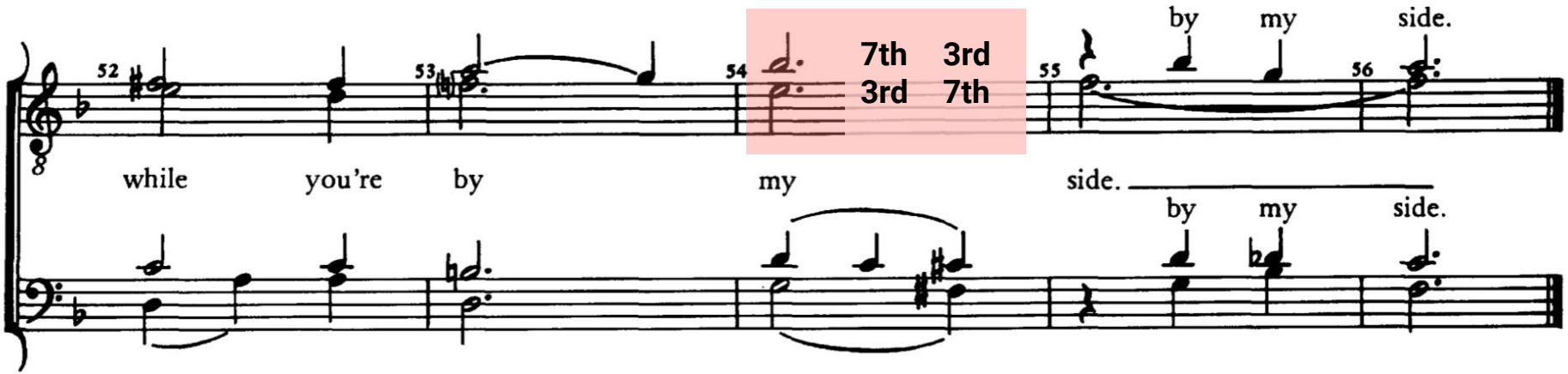


6. Sweet and Lovely - the solution!

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Sweet and Lovely - *The solution!*

Tenor:
+ 34 cents!!



C9 C7 F#7

Tuning by Numbers

Appendix: maths for working out cent adjustments

Why don’t *ET* and *JI* agree with one another?

- There are twelve semitones in an octave
- Let’s assign $f_1, f_2, f_3 \ldots f_{13}$ to be the frequencies of 13 notes separated by a semitone (eg *C* to *C*)



Why don’t *ET* and *JI* agree with one another?

2 : 1

Octave frequency ratio

$$\frac{f_{13}}{f_1} = 2$$

f_1 f_2 f_3 f_4 f_5 ... f_{13}



1:

$$\frac{f_{13}}{f_1} = 2$$

If we use a clever trick, this is the same as saying:

1a:

$$\frac{f_2}{f_1} \times \frac{f_3}{f_2} \times \frac{f_4}{f_3} \times \ \ldots \ \times \frac{f_{13}}{f_{12}} = 2$$

1a:

$$\frac{f_2}{f_1} \times \frac{f_3}{f_2} \times \frac{f_4}{f_3} \times \ \ldots \ \times \frac{f_{13}}{f_{12}} = 2$$

2:

From *ET*: $$\frac{f_2}{f_1} = \frac{f_3}{f_2} = \frac{f_4}{f_3} \ \ldots$$

1a:

$$\frac{f_2}{f_1} \times \frac{f_3}{f_2} \times \frac{f_4}{f_3} \times \ \ldots \ \times \frac{f_{13}}{f_{12}} = 2$$

2:

From *ET*: $\frac{f_2}{f_1} = \frac{f_3}{f_2} = \frac{f_4}{f_3} \ \ldots$

1a & 2:

$$\left(\frac{f_2}{f_1}\right)^{12} = 2$$

$$\frac{f_2}{f_1} = \sqrt[12]{2}$$

$$\frac{f_2}{f_1} = 1.0594630944$$

Now we can define a semitone (in *ET*) in terms of a ratio

$$s = \frac{f_2}{f_1} = 1.0594630944$$

Equal temperament

A perfect fifth is 7 semitones.

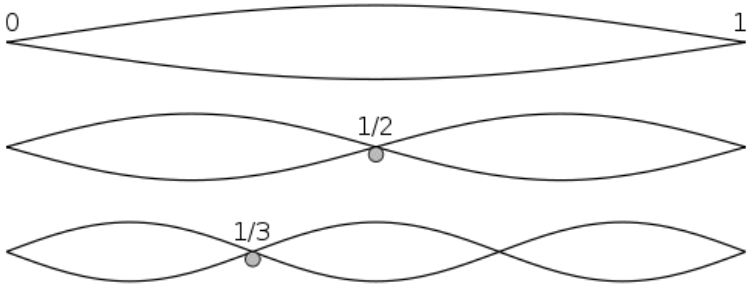
$$s^{7} = 1.4983070769$$



Just intonation

A perfect fifth is found between the 2nd and 3rd harmonics

$$2:3 \ or \ 3/2 = 1.5$$



What adjustments do we have to make to fix this?

Definition:

1 cent = 1/100th semitone

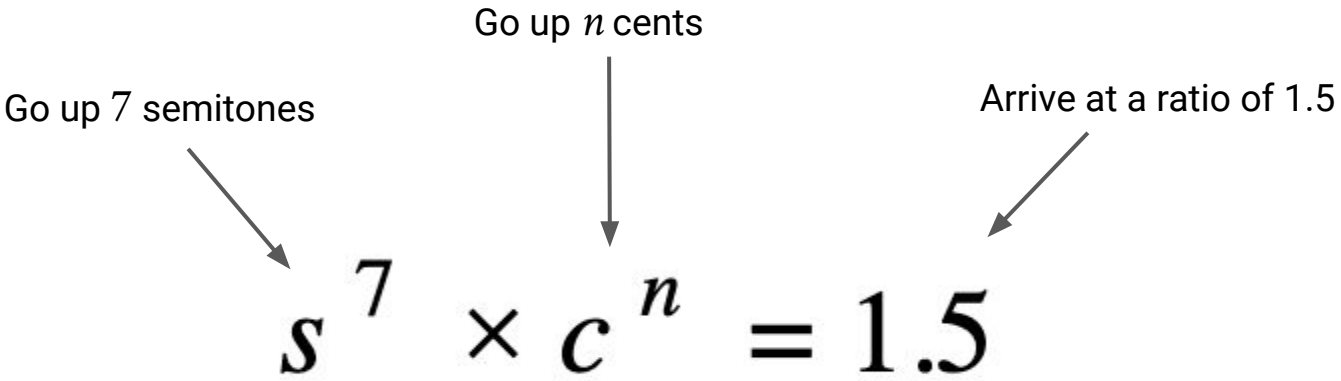
What adjustments do we have to make to fix this?

Definition:

1 cent = 1/100th semitone

$$1\ cent = c = \sqrt[100]{s} = 1.0005777895$$

What adjustments do we have to make to fix this?



All we need to do is find out what n is

$$s^7 \times c^n = 1.5$$

Scary maths!

$$\begin{cases} c^n = 1.5/s^7 \\ n = log_c(1.5/s^7) \\ n = 1.955... \end{cases}$$

→ $n \simeq 2$

In other words, we need to *raise* the (ET) note by two cents to make it agree with JI

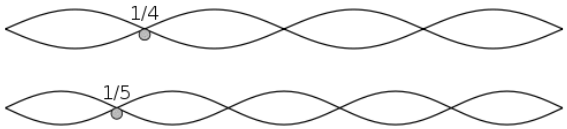
What about a major third?

Equal temperament

4 semitones



Just intonation



$$5 : 4 \;\; or \;\; 5/4 \;\; = \;\; 1.25$$

$$s^{4} \times c^{n} = 1.25$$

$$n = -13.69\ldots$$

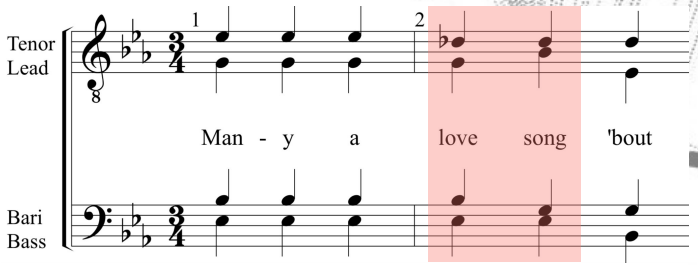
$$\rightarrow \quad n \simeq -14$$

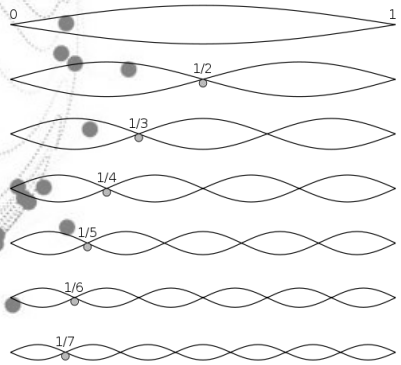
In other words, we need to *lower* the (ET) note by 14 cents to make it agree with JI

Sound Hypothesis *presents...*

Tuning by *Numbers*

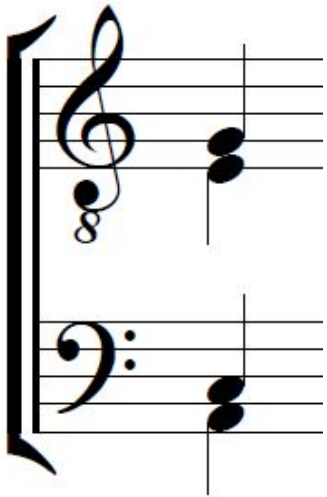






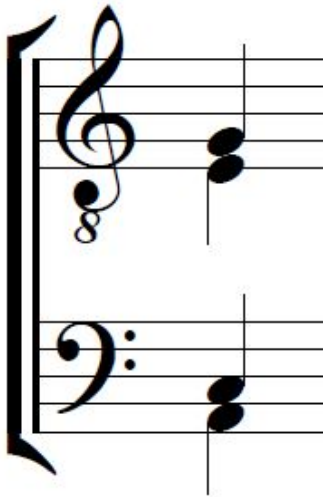
➔ The secrets of physics and maths in barbershop!

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-31
2

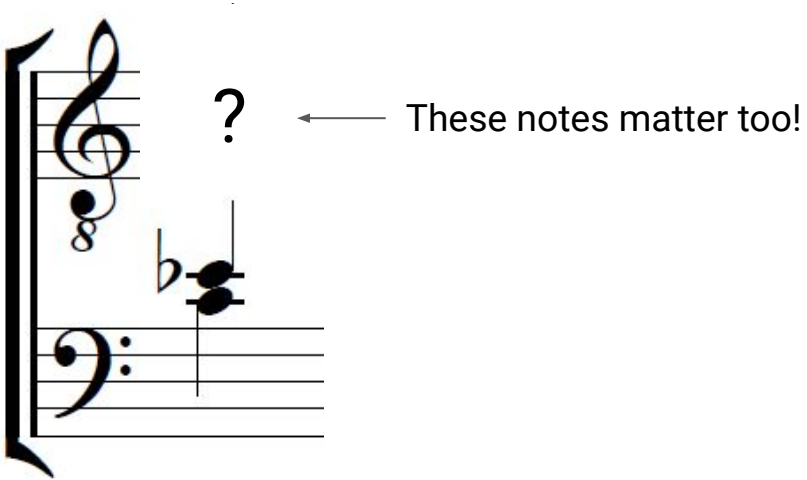
-33
0

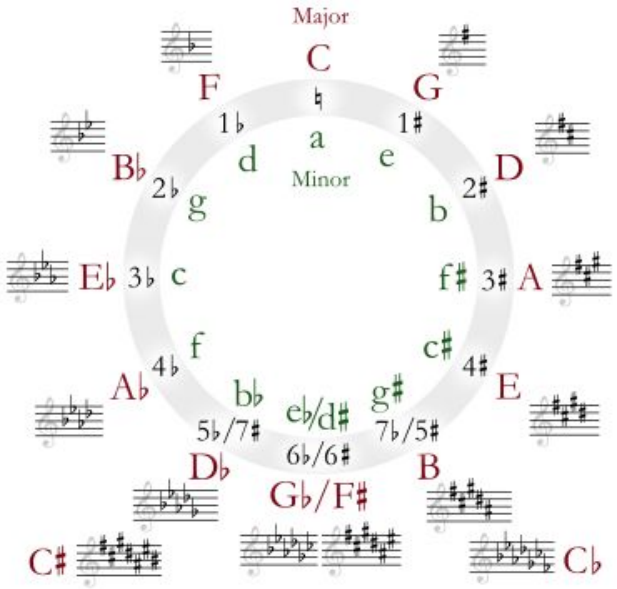


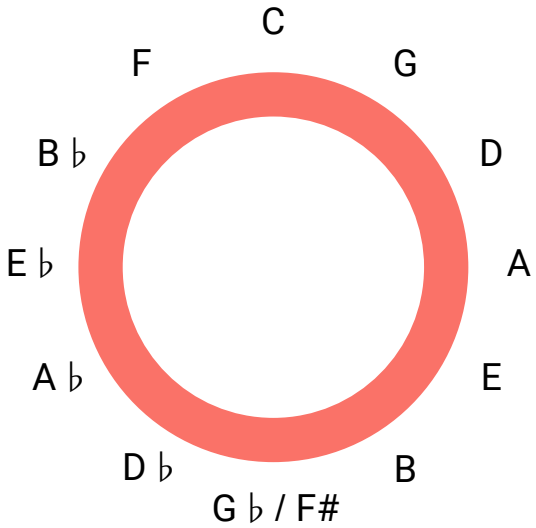
18
2

16
0

And we have to do this for all the notes of a chord *at the same time!*




Major
C
G
D
A
E
B
F♯/G♭
D♭
A♭
E♭
B♭
F
a
e
b
f♯
c♯
g♯
d♯/e♭
b♭
f
c
g
d
Minor
1♯
2♯
3♯
4♯
5♯/7♭
6♯/6♭
1♭
2♭
3♭
4♭
C♯
C♭


C
G
D
A
E
B
G♭ / F#
D♭
A♭
E♭
B♭
F